Robustness and Robust Optimization

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Minimax Variance

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In Huber’s case, the class of models is a location model in the neighborhood of a symmetric log-concave density and the functional $V(T, F)$ is the asymptotic variance of the estimator $T$ at model $F$. 

Huber’s Result in a simplistic Picture
Robust optimization generalizes the step from

$$ \inf_{\text{estimator } T} V(T, F) \text{ subject to } \mathbb{E}_\mu(T) = \mu $$

to

$$ \inf_{\text{estimator } T} \sup_{F \in \text{uncertainty set}} V(T, F) \text{ subject to } \mathbb{E}_\mu(T) = \mu. $$
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Finance: Risk v Reward

Let $w = (w_1, \ldots, w_k)$ be the parts of the portfolio invested in asset $i = 1, \ldots, k$ and let $X = (X_1, \ldots, X_k)$ be the returns of the assets over a fixed investment period. The portfolio is then

$$P = w^T X = w_1 X_1 + \ldots + w_k X_k.$$
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The mean and variance of the portfolio are \( \mu_P = w^T r \) (reward) and \( \sigma_P^2 = w^T \Sigma w \) (risk), where \( r = (r_1, \ldots, r_k) \) are the expected returns of the assets and \( \Sigma \) their covariance matrix.
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Markowitz considers portfolios that minimize the risk for a selected level of reward

\[
\min_w w^T \Sigma w \text{ subject to } w^T 1 = 1 \text{ and } w^T r \geq \mu_{\text{target}}
\]
The Solution in a Picture

By Parthiv.ravindran - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=16964880
Discussion of Markowitz

The Markowitz portfolio balances between the minimum risk portfolio

\[ w_{mv} = \Sigma^{-1} \mathbf{1} / (\mathbf{1} \Sigma^{-1} \mathbf{1}) \]

and the “maximum reward” portfolio

\[ w_{mr} = \Sigma^{-1} \mathbf{r} / (\mathbf{1}^T \Sigma^{-1} \mathbf{r}) . \]
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We obtain the same solution set if we consider

\[ \max_w \left( w^T r - \lambda \sqrt{w^T \Sigma w} \right) \text{ subject to } w^T 1 = 1 \]

where \( \lambda \geq 0 \) is a constant.
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We obtain the same solution set if we consider

\[ \max_w \left( w^Tr - \lambda \sqrt{w^T\Sigma w} \right) \text{ subject to } w^T1 = 1 \]

where \( \lambda \geq 0 \) is a constant. If we replace \( \lambda \) by \( -z_\alpha \), the \( \alpha \) quantile of the standard normal, we obtain for \( X \) multivariate normally distributed

\[ \max_w \left( \text{VaR}_\alpha(w^TX) \right) \text{ subject to } w^T1 = 1 . \]
The robust problem becomes

$$\max_w \min_{\text{Uncertainty set } r, \Sigma} \left( \text{VaR}_\alpha(w^T X) \right) \text{ subject to } w^T 1 = 1.$$
Introducing Uncertainty in $r$ and $\Sigma$

The robust problem becomes

$$\max_w \min_{r, \Sigma} \left( \text{VaR}_\alpha\left(w^T X\right) \right) \quad \text{subject to} \quad w^T 1 = 1.$$

Since the objective function

$$w^T r + z_\alpha \sqrt{w^T \Sigma w}$$

is linear in $r$, the minimization with regard to uncertainty in $r$ is easy.
The Robust Markowitz Problem

The Plot for two Assets with $(3 \pm 1, 1)$, $(4 \pm 1.8, 2)$

![Graph showing the correlation between two assets with different weight allocations and 5% VaR values.](image)

- **Correlation**: $-0.5$
- **5% VaR Values**: 0, 1, 2, 3, 4
- **Weight of First Asset**: 0.0 to 1.0

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The Plot for two Assets with \((3, 1 \pm 0.3), (4, 2 \pm 1)\)

\[
\begin{align*}
\text{Correlation} & \leq -0.5 \\
5\% \text{ VaR} & \geq 1.0, 1.5, 2.0, 2.5, 3.0 \\
w \text{of first asset} & \geq 0.0, 0.2, 0.4, 0.6, 0.8, 1.0
\end{align*}
\]

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The Robust Markowitz Problem

The Plot for two Assets with $(3, 1 \pm 0.3), (4, 2 \pm 1)$

Correlation $-0.5$

5% VaR

w of first asset

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Robust Portfolio Choice in a Complex Situation

- price of robustness
- price of non-robustness

percentage
price

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Conclusions

Replacing elaborate models by simpler ones combined with computable (convex) optimization algorithms is an alternative approach to reach robustness.
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- Thank you!